Readers' Forum

Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Remarks on a Comment by C.D. Bailey

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IN a recent Comment,[1] Professor Bailey made some observations about certain aspects of basic mechanics and fundamental mathematics. Those observations have motivated the following remarks.

Consider Professor Bailey's statement of the fundamental lemma of the calculus of variations. As presented, the lemma is false due to the failure to require continuity of $M(x)$ in the interval of consideration (x_1, x_2). This might be simply an oversight, but there are two other deficiencies of a far more serious conceptual nature.

First is the belief, which is evident throughout the Comment, that the fundamental lemma is valid only for $\eta(x)$ which vanish at the end points. This is not true. It is certainly true that most, if not all, statements of the lemma include the requirement that $\eta=0$ at the end points; but a close examination of the proof shows that this seeming requirement is, in fact, never used in the proof. It is also true that in many applications of the lemma, the admissible $\eta(x)$ do vanish at the end points; but this requirement is imposed by the mechanics of the problem (perhaps the principle of virtual work) or the analyst (through choice of admissible functions), *not* by the mathematics of the calculus of variations and the fundamental lemma.

This point can perhaps be strengthened by supposing that some textbook author presents his form of the fundamental lemma in which he "requires" that $\eta(x)$ vanish at the two ends and at the third points of the interval $[x_1, x_2]$. Such a lemma can be proved to be valid, but does this mean that the calculus of variations *requires* vanishing $\eta(x)$ at four points? Of course not. A whole host of additional seemingly required admissibility conditions can be included in the statement of a theorem; but if the essence of the theorem can be proved without the extra conditions, then it is faulty logic to claim that the theorem *requires* the conditions. This is exactly the situation with respect to vanishing $\eta(x)$ at the end points.

One cannot fault Professor Bailey for including the unnecessary admissibility requirement on η because he is in excellent company. However, the essence of his entire Comment is built around the claimed confusion caused by the supposedly *mathematical* requirement of the fundamental lemma for the vanishing of $\eta(x)$ at the end points. One might reasonably question the validity of observations which are based on an unnecessary requirement.

The next conceptual error appears when Professor Bailey states that "the function M must be some sort of 'balance' or 'conservation' equation which must be known to vanish, a priori,...." This is not true. The fundamental lemma is purely mathematical. There is no concept of energy balance or conservation attached to the lemma—none whatsoever. The science of mechanics sets the problem. The calculus of variations merely shows how to formulate variations, and the fundamental lemma merely describes the mathematical consequences of the mechanics-imposed requirement of a vanishing variation of a functional.

If the laws of mechanics require that $M(x)=0$, then it is certainly true (loosely speaking) that

$$\int_{x_1}^{x_2} M(x)\eta(x)\mathrm{d}x=0$$

for arbitrary $\eta(x)$. This has *nothing whatever* to do with the fundamental lemma of the calculus of variations. In other words, if the analyst has knowledge that $M=0$, then the fundamental lemma will probably never appear in a problem solution. Application of the lemma does *not* require knowledge that M must vanish in the domain. Quite the contrary. The lemma is used to *derive* the fact that $M=0$ in the domain, given that some functional must vanish for arbitrary $\eta(x)$.

Probably the most basic misconception in the Comment has to do with even mentioning the fundamental lemma of the calculus of variations. In his Comment, Professor Bailey is clearly beginning with Eq. (1) and ending with Eq. (3). Let us briefly go through the steps from Eq. (1) to Eq. (3) and see if the fundamental lemma ever appears.

In Eq. (1) as presented, the system is assumed to be in equilibrium. Here one begins with knowledge that $F-\dot{P}=0$, and Eq. (1) is obviously true. Note, however, that the equation has nothing whatever to do with the fundamental lemma. Indeed, the calculus of variations might never have been invented; all that is required are the fairly elementary facts that (0) $\delta r=0$ for every $\delta r(t)$ and

$$\int_{t_1}^{t_2} 0\mathrm{d}t=0$$

To go from Eq. (1) to Eq. (2), giving the so-called Hamilton's Law of Varying Action, requires only another fairly elementary operation of integration by parts. Still no appearance of the fundamental lemma or any other concepts of calculus of variations.

Now, mechanics has imposed the requirement that Eqs. (1) and (2) will be satisfied for every δr. Clearly, this requirement is not imposed by the calculus of variations or the fundamental lemma. Therefore, the analyst *chooses* to restrict consideration to those δr for which $\delta r=0$ at the end points, leading to Eq. (3), which can be associated with Hamilton's Principle.

The briefly outlined derivation given above, which is of course available in many texts (see, for example, Ref. 2) shows that the fundamental lemma never appears anywhere in Professor Bailey's development from Eq. (1) to Eq. (3). And yet he says at the end of Eq. (2), "If the Fundamental Lemma requires that the function, η, vanish at x_1 and x_2 *before* integration by parts, logically the functions δr_i must vanish at t_1 and t_2 *after* integration by parts. Thus, one obtains Hamilton's Principle...." It would appear that Professor Bailey believes that the fundamental lemma sets the

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requirement of $\delta r = 0$ at the ends; and because of the fundamental lemma one obtains Eq. (3). This is not true.

As shown in the derivation discussion, the fundamental lemma is not involved in deriving Eq. (3) from the principle of virtual work. Of course the fundamental lemma will be applied if one uses Hamilton's Principle to derive the differential equations of motion, but this most definitely in *not* what Professor Bailey does in his Comment. He is plainly attempting to say something about the derivation of Eqs. (2) and (3) from Eq. (1).

Professor Bailey states that Eq. (3) and "the concepts associated therewith have caused massive confusion with respect to direct solutions to the problems of mechanics and physics..." and later implies that this confusion has been caused by "foundational concepts of the Calculus of Variations...." But this cannot be true, because the calculus of variations has nothing whatever to do with the development of Eq. (3). How can we blame the fundamental lemma when it never appeared in the derivation?

With respect to Hamilton's Principle, the admissibility conditions have always been understood—the trial functions must take on specified values at the end points—and perhaps the best way to demonstrate that the calculus of variations causes no confusion is to actually display a suitable trial function. This can very easily be done in the form

$$q(t) = q_0\psi_0(t) + q_1\psi_1(t) + \sum_{n=1}^{\infty} a_n\phi_n(t)$$

where $q(t)$ is the system degree of freedom, $t=0$ and $t=t_1$ are the two end points in time, and $q=q_0$ and $q=q_1$ are the specified values at the end points.

The functions $\psi_i(t)$ and $\phi_i(t)$ are specified functions which satisfy the following conditions:

$$\psi_0(0) = 1, \qquad \psi_0(t_1) = 0$$

$$\psi_1(0) = 0, \qquad \psi_1(t_1) = 1$$

$$\phi_n(0) = 0, \qquad \phi_n(t_1) = 0 \qquad n = 1,2,...$$

This trial function can be used directly in an application of Hamilton's Principle, leading to an expression for the a_n in terms of q_0 and q_1. Then finally the magnitude of q_1 can be selected to provide satisfaction of the initial condition on velocity. A direct solution has been achieved through Hamilton's Principle with no confusion due to the calculus of variations.

Clearly, there has never been any need for confusion concerning Hamilton's Principle; and one can only speculate concerning the cause of any confusion which may have existed in the past. However, one can state with assurance that the difficulties did not arise from any foundational concepts of the calculus of variations; and it is certain that there has never been confusion among the knowledgeable involving the fundamental lemma and Hamilton's Principle.

References

[1]Bailey, C. D., "Comment on 'Finite Elements for Initial Value Problems in Dynamics'," *AIAA Journal,* Vol. 21, Jan. 1983, pp. 159-160.

[2]Meirovitch, L., *Analytical Methods in Vibrations,* Macmillan, New York, 1967, pp. 42-45.

Reply to Remarks by C. V. Smith Jr.

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ONCE again I must express my appreciation to Professor Smith for his comments. Smith sets the tone for his concerns when he writes,

1) "As presented, the lemma is false...." He then discusses additional concerns,

2) "...the fundamental lemma is not involved in deriving Eq. (3)...."

3) The apparent belief on my part "that the fundamental lemma is valid only for $\eta(x)$ which vanish at the end points."

4) "There is no concept of energy balance or conservation attached to the lemma...."

5) "Probably the most basic misconception...has to do with even mentioning the fundamental lemma...."

6) "...there has never been confusion...."

Each of Smith's concerns will be briefly addressed.

1) *"As presented, the lemma is false...."*

The lemma was taken from Ref. 1, p. 288, "The lemma has several formulations, one of which is that of Bolza:

Lemma: If M is a continuous function of x on $[x_1,x_2]$ and if

$$\int_{x_1}^{x_2} \eta M \mathrm{d}x = 0$$

for all functions η which vanish at x_1 and x_2 and possess a continuous derivative on $[x_1,x_2]$, then $M \equiv 0$ on $[x_1,x_2]$."

My omission of continuity on M in both Refs. 2 and 3 was inadvertent.

2) *"...The fundamental lemma is not involved in deriving Eq. (3)...."*

In Ref. 2, it is clearly stated that *my* starting point was *never* d'Alembert's principle nor any other force or energy balance equation. My starting point has always been, and will continue to be, a postulate[4] and the integral,

$$\int_{t_0}^{t_1} (T+W)\mathrm{d}t$$

(Refs. 4 and 5). The fact that I used Simkins'[6] starting point in Ref. 2 has nothing to do with my understanding of the meaning of the law of varying action and neither does what I find in the calculus of variations. My philosophy and my starting point in this response cannot be detailed due to space limitation. Instead, by invoking the simplest problem of the calculus of variations, I will demonstrate what I did in Ref. 2.

Given the functional,

$$\int_{x_1}^{x_2} F(y,y',x)\mathrm{d}x$$

which is assumed to satisfy the necessary requirements, find the first variation in accordance with the calculus of

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